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THE VISCOELASTICITY EQUATIONS OF AN ELASTOMER LAYER†

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A theory of an elastomer layer of linear viscoelasticity in the case of static and dynamic deformations is constructed. Some exact solutions of the problems of a plane layer are considered and their relation to layer theory is pointed out. The structure of the exact solution proves that it is possible to split the strained state into a fundamental state and a boundary layer. The equations of the layer give the fundamental state.

1. We will assume that the strain of a material is described by the Boltzmann–Volterra law of hereditary elasticity [1]

$$\sigma_{ii} = \tilde{K}e + 2\tilde{G}\left(e_{ii} - \frac{1}{3}e\right), \qquad \sigma_{ij} = \tilde{G}e_{ij} \tag{1.1}$$

$$\tilde{K} = K_0(1-K^*), \qquad \tilde{G} = G_0(1-G^*)$$

$$(K^*, G^*)\,u(t) = \int_{-\infty}^{t} [K'(t-\tau),\ G'(t-\tau)]\,u(\tau)\,d\tau \tag{1.2}$$

where K_0 and G_0 are the instantaneous moduli of elasticity, and K', G' are the kernels of the bulk and shear relaxation. The material is assumed to be non-uniform. The lower limit can be taken to be zero if $u = 0$ when $\tau < 0$.

The hereditary properties of elastomers (creep and relaxation) depend very much on the temperature. The time and temperature dependences are interrelated [1, 2]. At a variable temperature the behaviour of the material can be described by the same relations as for constant temperature, but with a changed time scale. The reduced time $d\xi = a(T)\,dt$ is introduced into (1.1) and (1.2) instead of t. The function $a(T)$ is usually taken in the form $\lg a(T) = -C_1(T-T_0)/(C_2 + T - T_0)$, where C_1 and C_2 are constants of the material [2], and T_0 and T are the initial and current temperature.

The fact that there is a temperature–time analogy enables prolonged tests to be carried out on a material for creep and relaxation under the usual conditions to be replaced by short-term tests at an increased temperature.

The Volterra integral operators (1.2) are bounded and difference operators, and they describe the behaviour of materials whose properties do not change with time. For elastomers, the mechanism of hereditary strain is distinct in the regions of the highly elastic and glass-like states and cannot be described by (1.1) and (1.2) with difference kernels over the whole range of temperature variation. The instantaneous moduli are obtained independently of the current temperature. This limits the range of application of the viscoelasticity law (1.1), (1.2) and the temperature–time superposition principle. No acceptable theory exists at present which reflects the change in the moduli with temperature on transferring from the rubber-like to the glass-like state. Empirical formulae do exist, however, one of which is given in [3]: $E(\xi) = E_\infty + (E_c - E_\infty)/(1 + \xi/t_R)^n$, where E_c and E_∞ are the short-term (at the vitrification temperature) and the long-term moduli, t_R is the characteristic relaxation time, and usually $n = 0.3$.

2. We will consider static strain. According to the Volterra principle, in order to obtain a solution of the viscoelastic problem we can initially construct a solution of the elastic problem and then replace the moduli of elasticity by operators in the final formulae. We will use the results of elastic-layer theory [4, 5] below.

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The equations of elasticity for a thin elastomer layer contain two small parameters: a geometrical parameter $\varepsilon = h/R$, the ratio of the characteristic dimensions and the physical dimensions, and $1 - 2\nu = G/K$, the ratio of the shear modulus and the bulk compression modulus. We will assume that $1 - 2\nu \sim \varepsilon^q$, $q > 0$. Corresponding to three cases of interest for layer theory, namely, $q = 1$, 2 and 3, a layer will be called very thin, thin or of medium thickness. If q is considerably less than unity, the model of an infinitely extended layer is applicable; if q is considerably greater than three, the model of an incompressible material is applicable. The equations of layer theory in the zeroth approximation are identical in all three cases. To fix our ideas we will henceforth take $q = 2$.

We introduce orthogonal curvilinear coordinates (α, β, z) where $(\alpha, \beta) \in S$, $|z| \leqslant h/2$, S is the median surface of the layer and h is the thickness. The position of a point is specified by the vector $\mathbf{R} = \mathbf{r}(\alpha, \beta) + z\mathbf{n}$, $\mathbf{n} \perp S$.

In the zeroth approximation in ε we obtain [5]

$$\mathbf{U} = \left(\frac{1}{2} + \zeta\right)\mathbf{U}^+ + \left(\frac{1}{2} - \zeta\right)\mathbf{U}^- + $$
$$+ \frac{1}{8}(1 - 4\zeta^2)h\left[h\frac{\nabla Ke}{G} - \left(\frac{1}{3}\zeta h^2 \operatorname{div}_S \frac{\nabla Ke}{G} - \operatorname{div}_S(\mathbf{U}^+ - \mathbf{U}^-)\right)\mathbf{n}\right] \tag{2.1}$$

$$h^2 \operatorname{div}_S \frac{\nabla Ke}{G} - 12e = -12\frac{W^+ - W^-}{h} - 6\operatorname{div}_S(\mathbf{U}^+ + \mathbf{U}^-) \tag{2.2}$$

where $\mathbf{U}$ is the displacement vector of a point, $e = \operatorname{div}\mathbf{U}$ is the relative increment in the volume, $\mathbf{U}^+$ and $\mathbf{U}^-$ are the displacements of the faces, and W is the projection of the vector $\mathbf{U}$ on to the normal $\mathbf{n}$. The operations div_S and ∇ are carried out on the median surface S. In (2.1) and (2.2) we assume that the moduli of elasticity K and G are independent of z and $\zeta = z/h$.

The solution of the boundary-value problems of the layer reduces to integrating Eq. (2.2) for the function e with the boundary condition $Ke = p$, where p is the pressure. Under kinematic conditions on the faces of the layer the vectors $\mathbf{U}^+$ and $\mathbf{U}^-$ are specified, but under other types of conditions they are regarded as unknowns.

Equations (2.1) and (2.2) give the elastic solution. In order to transfer to the viscoelastic solution we have to replace the moduli K and G by operators.

3. We will obtain the exact solutions of a number of problems of a plane layer with rigid faces. The material is assumed to be uniform here. We will use Cartesian coordinates.

In the problem of the tension–compression and bending of a layer the boundary conditions on the faces $z = \pm h/2$ have the form

$$(U_x, U_y) = 0, \quad W = \pm(a_z - x\omega_y + y\omega_x)/2 \tag{3.1}$$

where a_z, ω_x, ω_y are the relative displacement and rotations.

We will obtain the solution of the equations of equilibrium in displacements $\operatorname{grad} e + (1 - 2\nu)\Delta\mathbf{U} = 0$ by the method of homogeneous Papkovich–Lur'ye solutions

$$(U_x, U_y) = \frac{h}{\lambda}\varphi(\lambda, \zeta)(\Phi'_x, \Phi'_y) + (1 - 4\zeta^2)\frac{h}{8(1 - 2\nu)}(-\omega_y, \omega_x) \tag{3.2}$$
$$W = \psi(\lambda, \zeta)\Phi + \zeta(W^+ - W^-), \quad he = 4(1 - 2\nu)\Phi\cos\lambda\zeta + W^+ - W^-$$
$$\varphi = \operatorname{tg}\tfrac{1}{2}\lambda\cos\lambda\zeta - 2\zeta\sin\lambda\zeta, \quad \psi = \operatorname{ctg}\tfrac{1}{2}\lambda\sin\lambda\zeta - 2\zeta\cos\lambda\zeta$$

The functions $\Phi(\lambda, x, y)$ and the parameters λ are found from the equations

$$h^2\Delta\Phi - \lambda^2\Phi = 0, \quad (3 - 4\nu)\sin\lambda - \lambda = 0 \tag{3.3}$$

In (3.2) the summation is carried out over λ.

The boundary conditions for the first equation of (3.3) are found in a special way, in order that the displacements (3.2) satisfy the specified static conditions on the side surface of the layer.

A consequence of the low compressibility of the material is the presence of two small roots of the second equation of (3.3) $\lambda = \pm\sqrt{(12(1 - 2\nu))}$, which correspond to the values of the Papkovich functions $\varphi = \pm(1 - 4\zeta^2)\sqrt{(3(1 - 2\nu))}$, $\psi = -2\zeta(1 - 4\zeta^2)(1 - 2\nu)$.

For small roots λ the function Φ is constant over the thickness of the layer with an error of $1 - 2\nu$, and it then follows from the first equation of (3.3) and formulae (3.2) that

$$h^2\Delta e - 12(1-2\nu)e = -12(1-2\nu)(W^+ - W^-)/h \tag{3.4}$$

Formulae (3.2) for small λ and Eq. (3.4) are completely identical with the results of the analogous problem of layer theory (2.1), (2.2).

Equation (3.3) has two small real roots λ; the remaining roots are complex and are not small. If the layer is thin, the solution will be of the boundary-layer type. Hence, for a thin layer its stress–strain state can be split into a fundamental layer, corresponding to small roots λ, and a boundary layer. In the zeroth approximation, the boundary layer has no effect on the fundamental slowly varying state.

In the problem of the shear of a layer, the boundary conditions on the faces have the following form: $U_x = \pm a_x/2$, $U_y = \pm a_y/2$, $W = 0$, where a_x and a_y are the relative displacements of the base. The solution of the boundary-value problem is as follows:

$$(U_x, U_y) = h\lambda^{-1}\psi(\lambda,\zeta)(\Phi'_x, \Phi'_y) + \zeta(a_x, a_y)$$
$$W = -\varphi(\lambda,\zeta)\Phi, \qquad he = -4(1-2\nu)\Phi\cos\lambda\zeta \tag{3.5}$$

The function Φ is found from the first equation of (3.3), while the parameter λ satisfies the equation $(3-4\nu)\sin\lambda + \lambda = 0$. Here all the roots are complex and are not small, and the solution corresponding to them will be of the boundary-layer type. The fundamental state is given by the particular solution in (3.5), i.e. it is simple shear.

We presented the solutions of the problems of elasticity above; for viscoelasticity the parameters ν and λ must be replaced by the operators $\tilde{\nu} = \nu_0(1 + \nu^*)$, $\tilde{\lambda} = \lambda_0(1 - \lambda^*)$. The limits of variation of the quantity $\tilde{\nu} \cdot 1$ (the convolution of the operator $\tilde{\nu}$ with unity [1]) are small: $\nu_0 \leqslant \tilde{\nu} \cdot 1 < 0.5$, since the initial value of Poisson's ratio ν_0 is close to 0.5.

4. We will now consider the dynamic problem. The boundary conditions on the surfaces of the layer are specified to be of the same type as in the static problem. We are also given the initial conditions—the displacements and velocities of the points.

In addition to the small parameters mentioned above, the equations of motion will also contain parameters of the frequency of the velocity (or oscillations), which may vary over a wide range. We will say more below about the limitations on the value of the frequency used in deriving the equations of the layer.

The procedure for constructing the dynamic theory of the layer using the asymptotic method is well known [6]. We will use this method for the case of viscoelasticity. The equations of motion, written in displacements, in the zeroth approximation in $\varepsilon = h/R$ (as previously, we assume $1 - 2\nu \sim \varepsilon^2$), have the form

$$A^{-1}(\tilde{K}e)'_\alpha + (\tilde{G}U'_z)'_z - \tilde{\rho}U''_t = 0$$
$$B^{-1}(\tilde{K}e)'_\beta + (\tilde{G}V'_z)'_z - \tilde{\rho}V''_t = 0, \qquad (\tilde{K}e)'_z = 0 \tag{4.1}$$

where A and B are the Lamé parameters of the system of coordinates (α, β) on S, and $\tilde{\rho}$ is the density operator, similar to (1.2). The required functions are the variables U, V and W and the relative increment of the volume.

In the zeroth approximation in ε we have

$$e = W'_z + [(BU)'_\alpha + (AV)'_\beta]/AB \tag{4.2}$$

For Eqs (4.1) and (4.2) on the faces of the layer $z = \pm h/2$ the previous conditions remain the same, but on the side surface we can specify only one asymptotically principal condition. When $t = 0$ the displacements U and V are given as well as their velocities as a function of the variable z.

For unsteady loading we can apply a Laplace transformation with respect to time to Eqs (4.1) and (4.2) and then integrate with respect to the variable z, as in the static problem [4, 5]. Below we will consider the problem of the oscillations of the layer when it is excited harmonically. Its elastic solution is given in [6].

Periodic stresses correspond to time-periodic strains in (1.1) and (1.2). If the lower limit in (1.2) is taken to be zero, the stresses become non-periodic. However, for kernels which satisfy the decaying memory condition, which disturb the periodicity, the integral term approaches zero as t increases.

We will introduce the complex displacements

$$(U, V, W) = (\overline{U}, \overline{V}, \overline{W})\, e^{i\omega t} \tag{4.3}$$

The strains e_{ij} and the functions e have the same time-dependence. The formulae for calculating the stresses are as follows [1]:

$$\sigma_{ii} = \overline{K}e + 2\overline{G}\left(e_{ii} - \frac{1}{3}e\right), \quad \sigma_{ij} = \overline{G}e_{ij} \tag{4.4}$$

$$\overline{K} = K_0\left[1 - \int_0^\infty K'(x)\, e^{-i\omega x} dx\right], \quad \overline{G} = G_0\left[1 - \int_0^\infty G'(x)\, e^{-i\omega x} dx\right]$$

where $\overline{K} = K_1 + iK_2$, $\overline{G} = G_1 + iG_2$ are complex moduli of elasticity which depend on the frequency ω.

The quantities K_1 and G_1 represent the elastic deformation energy, while K_2 and G_2 represent the dissipated energy. The mechanical loss tangents for bulk and shear strains can be calculated from the equations $\operatorname{tg} \varphi_K = K_2/K_1$, $\operatorname{tg} \varphi_G = G_2/G_1$.

The approximate formulae for the stresses have the form

$$\sigma_{ii} = \sigma = \overline{K}e, \quad \sigma_{13} = \overline{G}U_z', \quad \sigma_{23} = \overline{G}V_z', \quad \sigma_{12} = 0$$

Using (4.3) and (4.4) we can convert Eqs (4.1) to the form

$$\begin{gathered} A^{-1}(\overline{K}e)_\alpha' + (\overline{G}U_z')_z' + \overline{\rho}\omega^2 U = 0 \\ B^{-1}(\overline{K}e)_\beta' + (\overline{G}V_z')_z' + \overline{\rho}\omega^2 V = 0, \quad (\overline{K}e)_z' = 0 \end{gathered} \tag{4.5}$$

Suppose the moduli $\overline{K}$ and $\overline{G}$ are independent of z. We integrate Eqs (4.5) and (4.2) by the method of separation of variables. As a result we obtain the complex displacements and an equation for the function e

$$U = a_+U^+ + a_-U^- - b_0(\overline{K}e)_\alpha' / (A\overline{\rho}\omega^2), \quad V = a_+V^+ + a_-V^- - b_0(\overline{K}e)_\beta' / (B\overline{\rho}\omega^2)$$

$$W = \left(\frac{1}{2} + \zeta\right)W^+ + \left(\frac{1}{2} - \zeta\right)W^- + \frac{h}{\overline{\rho}\omega^2}\operatorname{div}_S(c\nabla\overline{K}e) + \frac{h}{2}\operatorname{div}_S[(c+d)\,\mathbf{U}^+ + (c-d)\,\mathbf{U}^-] \tag{4.6}$$

$$\frac{1}{\overline{\rho}\omega^2}\operatorname{div}_S[(2f-1)\nabla\overline{K}e] - e = -\frac{1}{h}(W^+ - W^-) - \operatorname{div}_S f(\mathbf{U}^+ + \mathbf{U}^-); \quad e\big|_{\partial S} = 0 \tag{4.7}$$

$$a_\pm = \frac{\sin k(\frac{1}{2} \pm \zeta)}{\sin k}, \quad b_0 = 1 - \frac{\cos k\zeta}{\cos \frac{1}{2}k}$$

$$c = \zeta\frac{2}{k}\operatorname{tg}\frac{k}{2} - \frac{\sin k\zeta}{k\cos\frac{1}{2}k}, \quad d = \frac{1}{k}\operatorname{ctg}\frac{k}{2} - \frac{\cos k\zeta}{k\sin\frac{1}{2}k}$$

$$f = \frac{1}{k}\operatorname{tg}\frac{k}{2}, \quad k = h\omega\sqrt{\frac{\overline{\rho}}{\overline{G}}}$$

where $\overline{\rho}$ is the complex density and k is a complex parameter.

If we assume that all the functions and parameters in (4.6) and (4.7) are real, we obtain the relations of the dynamic theory of an elastic layer [6]. However, there is no complete analogy between the viscoelastic and the elastic problems since there is a phase shift between the oscillation of the strains and the stresses.

Equations (4.1) describe shear waves propagating in the surfaces z = const. For an elastomer the ratio of the velocities of transverse and longitudinal waves is small (of the order of $\sqrt{(1-2\nu)}$). The time taken for a longitudinal wave to traverse the region S is of the same order as the time taken for the transverse wave to traverse the thickness of the layer. These wave processes do not occur in Eqs (4.1).

When deriving the equations of the layer we assumed that k is of the order of unity. When $k = \pi$ the displacements (4.6) become infinite and the coefficient of the leading derivatives in Eq. (4.7) vanish. Hence, the limit of applicability of the theory of the layer with respect to frequency will be taken as $k < \pi$. The frequency $\omega = \pi b/h$ corresponding to $k = \pi$ will be the lowest natural frequency of oscillations of the elastic layer (b is the velocity of shear waves). The dynamic equations (4.6) and (4.7) reduce to the equations of the static problem (2.1), (2.2) as $\omega \to 0$.

We will calculate the irreversible part of the work of deformation. We must use real functions here. We will put

$$(U, V, W) = (U_1, V_1, W_1)\sin\omega t + (U_2, V_2, W_2)\cos\omega t$$

Using (1.1) and (1.2) we obtain the stresses

$$\sigma_{ii} = \left[K_1 e_1 - K_2 e_2 + 2G_1\left(e_{ii}^1 - \frac{1}{3}e_1\right) - 2G_2\left(e_{ii}^2 - \frac{1}{3}e_2\right)\right]\sin\omega t +$$
$$+\left[K_1 e_2 + K_2 e_1 + 2G_1\left(e_{ii}^2 - \frac{1}{3}e_2\right) + 2G_2\left(e_{ii}^1 - \frac{1}{3}e_1\right)\right]\cos\omega t$$
$$\sigma_{ij} = (G_1 e_{ij}^1 - G_2 e_{ij}^2)\sin\omega t + (G_1 e_{ij}^2 + G_2 e_{ij}^1)\cos\omega t$$

The specific work of deformation in a time t is equal to

$$A = \int_0^t (\sigma_{11}\dot{e}_{11} + \ldots + \sigma_{23}\dot{e}_{23})\,dt \approx \int_0^t (\sigma\dot{e} + \sigma_{13}\dot{e}_{13} + \sigma_{23}\dot{e}_{23})\,dt =$$
$$= \frac{1}{2}(\sigma e + \sigma_{13}e_{13} + \sigma_{23}e_{23})\Big|_0^t + Dt$$
$$D = \tfrac{1}{2}\omega[K_2(e_1^2 + e_2^2) + G_2(U_{1,z}'^2 + V_{1,z}'^2 + U_{2,z}'^2 + V_{2,z}'^2)]$$

The first term is a periodic function of time and is the reversible part of the work, the second term is proportional to time and is the part of the work which is dissipated, and D is the value of the irreversible part of the work per unit time, which is called the dissipation power. The expression for D can be used as a function of the heat sources in the heat-conduction equation when solving the problem of the dissipative heating up of an elastomer layer.

5. We will consider the exact solutions of some problems of viscoelasticity when a plane layer is excited harmonically. The side surface is free, and the material is homogeneous. The equations and their solutions are written below in complex form. We will seek a solution of the equations of motion

$$\bar{b}^2[\operatorname{grad} e + (1 - 2\bar{\nu})\Delta \mathbf{U}] + (1 - 2\bar{\nu})\omega^2\mathbf{U} = 0 \tag{5.1}$$

The torsion of a ring layer. We will use cylindrical coordinates. The boundary conditions on the faces $z = \pm h/2$: $U_r = W = 0$, $U_\varphi = \pm r\omega_z/2$, where ω_z is the angle of relative rotation of the surfaces. The solution of the problem is $U_\varphi = \tfrac{1}{2}r\omega_z \sin k\zeta/\sin \tfrac{1}{2}k$, $k = h\omega/b$. The displacements U_r and W and the function e are equal to zero.

Shear of a layer. The boundary conditions on the faces are similar to those in the static problem. We will write the solution of Eqs (5.1) in the form

$$(U_x, U_y) = \frac{h}{\lambda}\psi(\Phi_x', \Phi_y') + \frac{\sin k\zeta}{2\sin\tfrac{1}{2}k}(a_x, a_y), \quad W = \frac{k_1}{\lambda}\varphi\Phi \tag{5.2}$$
$$\varphi = \cos k_1\zeta - \frac{\cos\tfrac{1}{2}k_1}{\cos\tfrac{1}{2}k_2}\cos k_2\zeta, \quad \psi = \sin k_1\zeta - \frac{\sin\tfrac{1}{2}k_1}{\sin\tfrac{1}{2}k_2}\sin k_2\zeta$$
$$k^2 = h^2\omega^2/\bar{b}^2, \quad k_1^2 = \lambda^2 + h^2\omega^2/\bar{a}^2, \quad k_2^2 = \lambda^2 + h^2\omega^2/\bar{b}^2$$

where $\bar{a}$ and $\bar{b}$ are the complex velocities of the longitudinal and transverse waves.

The functions Φ are solutions of Eqs (3.3), while the characteristic numbers λ are found from the equation

$$\lambda^2 \operatorname{tg}\tfrac{1}{2}k_1 = k_1 k_2 \operatorname{tg}\tfrac{1}{2}k_2 \tag{5.3}$$

In (5.2) we have assumed summation over λ.

The problem of tension–compression and bending of a layer. Conditions of the same type as in the static problem (3.1) are specified on the faces $z = \pm h/2$. From Eq. (5.1) using the method of homogeneous solutions, we obtain

$$(U_x, U_y) = \frac{h}{\lambda}\varphi(\Phi'_x, \Phi'_y) + \frac{h\varphi_0}{2k_1^0 \sin \frac{1}{2}k_1^0}(\omega_y, -\omega_x) \tag{5.4}$$

$$W = -\frac{k_1}{\lambda}\varphi\Phi + \frac{\sin k_1^0 \zeta}{2\sin \frac{1}{2}k_1^0}(W^+ - W^-)$$

where k_1^0 and φ_0 are the values of k_1 and φ when $\lambda = 0$.

The functions Φ are the solutions of Eqs (3.3), while λ are the roots of the equation which differs from (5.3) by the replacement $k_1 \leftrightarrow k_2$.

In view of the smallness of the ratio of the velocities of the shear and bulk strains this equation has small roots

$$\lambda^2 = (1 - 2\bar{v})\, k^2 / (2k^{-1} \operatorname{tg} \tfrac{1}{2}k - 1), \quad k = h\omega / \bar{b}$$

The results of layer theory (4.6) and the accurate solution for small λ are identical with an error of $1 - 2v$.

Solutions of elastic problems can be obtained from (5.2) and (5.4) if the elasticity parameters are assumed to be real. The values of the parameter $\lambda^2 \to 12(1 - 2v)$ as $\omega \to 0$.

6. We will consider two problems of elasticity of the harmonic oscillations of a cylindrical hinge which illustrate the possibilities of dynamic layer theory. Exact and approximate solutions are obtained for these problems using layer theory, and the results are compared.

The torsion of a cylindrical hinge (plane strain). Suppose r_1 and r_2 are the inner and outer radii and h is the thickness of the layer. The boundary conditions on the faces are as follows: when $r = r_1$, $U_\varphi = r_1\theta^-$ and when $r = r_2$, $U_\varphi = r_2\theta^+$, where U_φ and θ are the circumferential displacement and the angle of rotation (amplitude values of the functions are employed). The solution of the plane elasticity problem has the form

$$\theta = \tfrac{1}{2}\omega b^{-1}(A_1 J_0 + A_2 N_0), \quad U_\varphi = A_1 J_1 + A_2 N_1$$

where J_0, N_0, J_1 and N_1 are Bessel and Hankel functions of argument $r\omega b^{-1}$.

Determining the constants A_1 and A_2 from the boundary conditions and calculating the torque we obtain the stiffness relations

$$M^+ = d_{11}\theta^+ + d_{12}\theta^-, \quad M^- = d_{21}\theta^+ + d_{22}\theta^-$$

We will not derive the exact values of the dynamic stiffness coefficients, but layer theory gives

$$d_{11} = d_{22} = 2\pi G R^3 h^{-1} k \operatorname{ctg} k, \quad d_{12} = d_{21} = -2\pi G R^3 h^{-1} k / \sin k$$

where R is the mean radius and $k = h\omega b^{-1}$.

We calculated the stiffnesses for different frequencies using the exact and approximate formulae for a layer with the following parameters: $r_1 = 49.5$ cm, $r_2 = 50.5$ cm, $h = 1$ cm, $G = 10$ kg/cm^2, $K = 25 \times 10^3$ kg/cm^2 and $\rho = 1$ g/cm^3. The velocities of transverse and longitudinal waves were $b = 3.13 \times 10^3$ cm/s and $a = 1.56 \times 10^5$ cm/s.

The frequency $\omega_0 = \pi b/h = 9850$ rad/s, which is the lowest natural frequency of oscillations, corresponds to the limit of applicability of layer theory. For frequencies $\omega < \omega_0$ the agreement between the dynamic stiffnesses using the exact and approximate solutions is completely satisfactory (at a frequency $\omega = 9000$ rad/s the error is less than 10%).

Radial shear of a cylindrical hinge. The boundary conditions on the faces are as follows: for $r = r_1$, $U_r = a_x^- \cos\varphi$, $U_\varphi = -a_x^- \sin\varphi$ and for $r = r_2$, $U_r = a_x^+ \cos\varphi$, $U_\varphi = -a_x^+ \sin\varphi$; (r, φ) are polar coordinates.

The exact solution of the plane problem for amplitude values of the functions has the form

$$e = A_1 J_1(\alpha) + A_2 N_1(\alpha), \quad \theta = B_1 J_1(\beta) + B_2 N_1(\beta)$$

$$U_r = -\frac{r}{\alpha} e'_\alpha + \frac{r}{\beta^2}\theta, \quad U_\varphi = -\frac{r}{\alpha^2} e - \frac{r}{\beta}\theta'_\beta; \quad \alpha = \frac{r\omega}{a}, \quad \beta = \frac{r\omega}{b}$$

The dynamic stiffness relations are as follows:

$$F_x^+ = d_{11} a_x^+ + d_{12} a_x^-, \quad F_x^- = d_{21} a_x^+ + d_{22} a_x^-$$

In view of their complexity we will not give the exact values of the dynamic stiffnesses, but for layer theory these relations have the form

$$F_x^+ = -F_x^- = \pi K R\left[1 - \frac{\varepsilon}{2}(2f-1)\right]\left[1 + \frac{\varepsilon^2}{1-2\nu}\frac{1}{k^2}(2f-1)\right]^{-1}\frac{a_x^+ - a_x^-}{h}$$

For a layer with the above parameters we calculated the dynamic stiffnesses, the relative increment in the volume e and the angle of rotation θ for different frequencies ω. The agreement with the accurate solution was very good up to the critical frequencies ω_0. We confirmed that the distribution of the function e was close to constant over the thickness of the layer, and also the law of the distribution of the angle of rotation θ given by layer theory.

The problems considered differ in the nature of the strain in the layer: in the first there were only shear strains, while in the second there were both shear and bulk strains. An analysis of the problems confirms the correctness of the hypotheses used in deriving the equations of the layer and the estimate of the frequency limits of its applicability. It follows from the numerical results that one cannot replace the dynamic stiffnesses by the static stiffnesses as is often done for elastomer shock absorbers.

The simplicity with which solutions of the boundary-value problems are obtained using layer theory should be noted. The solutions of similar viscoelastic problems can be obtained from the elastic ones if we assume that the moduli of elasticity and the other parameters are complex.

7. Thus, we have obtained the dynamic equations of a curvilinear layer of non-uniform elastomer material for the problems of elasticity and viscoelasticity. In the case of plane-layer problems we have shown their connection with the exact solutions and we have established the physical basis. By setting up a layer theory we have been able to reduce the dimensions of the boundary-value problems and to eliminate the problems involved in the low compressibility of the material when solving them. The aims of setting up a layer theory and its value are similar to the theories of shells of the plates. The latter is constructed for static conditions on the faces of the body, while layer theory is set up for kinematic or mixed conditions. The two-dimensional equations of these theories differ qualitatively. Whereas in the classical theory of shells one must solve eighth-order equations, in layer theory one only needs to solve a single second-order equation.

The main applications of the layer equations are to solve problems of stability and the dynamics of multilayer rubber–metal components, which are widely used in technology as elastic hinges, shock absorbers, vibration-protection systems, etc. In these structures the deformation of the rubber layers is constrained by the boundary conditions on the faces, since the metal layers cannot have large deformations, and their bending is limited by the conditions at the bases of the multilayer system. Hence, the linear layer theory considered in this paper has some practical value. It is extremely important to set up a non-linear theory.

Special investigations are also required of the following problems of the dynamic theory of a layer and structures: the frequency limits of applicability of layer theory, the roots λ of the characteristic equations at different frequencies, an analysis of the natural and forced oscillations of the layer and of structures, the motion of a mass on a multilayer rubber–metal shock absorber, etc.

Similar problems of the dynamic theory of plates were considered in [7], where a rigorous solution was obtained of the dynamic problem of elasticity, the equations of the oscillations of thin plates were established, and the limits of their applicability were determined. It is interesting that the limits of applicability of the dynamic equations of a layer and a plate turn out to be practically the same ($\varepsilon = h/R < 1$, $\omega < \pi b/h$).

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